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# Independent SU(2)-loop variables and the reduced configuration space of SU(2)-lattice gauge theory

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We give a reduction procedure for SU(2)-trace variables and an explicit description of the reduced configuration space of pure SU(2)-gauge theory on the hypercubic lattices in two, three and four dimensions, using an independent subset of the gauge-invariant Wilson loops.

## 1. Introduction

During the last five years a remarkable unification in the description of Yang–Mills theory and gravity has been taking place. This was made possible by Ashtekar’s introduction of gauge theory like variables for canonical gravity, namely, a connection one-form  $A$  and its canonically conjugate “electric field”  $E$ , with internal SU(2)<sub>C</sub>-degrees of freedom [1]. This way the phase space of pure gravity becomes imbedded into the phase space of a gauge theory with gauge group SU(2)<sub>C</sub>  $\simeq$  SL(2, C). As a consequence, methods of gauge theory have been applied within the new formulation. In particular, Jacobson and Smolin were the first to use holonomies, i.e. non-local variables depending on closed curves in three-space, to describe pure gravity [2]. Such loop variables were employed subsequently by Rovelli and Smolin [3] in their proposal for a new non-perturbative quantization of canonical gravity. (A review of this and other results may be found in reference [4].) Their philosophy is to use only gauge-invariant loop variables to parametrize the reduced (with respect to the Gauss law constraints) phase space and quantize the theory by finding a representation of the classical Poisson algebra formed by these variables. This has in turn led to renewed interest in a similar loop formulation of classical and quantum Yang–Mills theory [5–9].

In this paper we will be exclusively concerned with the classical loop variables which feature prominently in these loop approaches and are of the form

$$T(\alpha) := \text{Tr } P \exp \oint_{\alpha} A_a(\alpha(t)) \dot{\alpha}^a(t) dt, \quad (1.1)$$

with  $P$  denoting path-ordering and  $\alpha$  either a closed curve on some three-mani-

fold or a closed path on a hypercubic lattice. The connection one-form  $A$  takes values in the gauge algebra  $\mathfrak{su}(2)$ , or its complexified version  $\mathfrak{su}(2)_{\mathbb{C}}$  in the gravitational case. Although for simplicity suppressed in the notation, it is understood that  $T(\alpha)$  depends on the gauge potential  $A$ . In the context of Yang–Mills theory, both on the lattice and in the continuum, the quantities  $T(\alpha)$ , sometimes called Wilson loops, are well known and have been used in various formulations to describe part of the theory (see, for example, refs. [10–12]). The main properties of the  $T(\alpha)$  are their invariance under gauge transformations of the  $A$ ’s and independence of the base point of the loop  $\alpha$  which can easily be demonstrated. Moreover, unlike local variables, they can carry diffeomorphism-invariant information which is important in all applications to gravity.

My intention is to develop further a loop approach that is exclusively based on the use of gauge-invariant variables of type (1.1), and is thus in line with some recent proposals for the quantization of both gauge theory and gravity [3,5,7,8]. This relies on the well-known fact that any gauge-invariant function of the connection  $A$  in a pure  $SU(N)$  gauge theory can be expressed in terms of the traces of holonomies of closed curves, i.e. the Wilson loops  $T(\alpha)$ . More precisely, there is a one-to-one correspondence between configurations  $A(x)$  modulo gauge transformations (i.e. elements of the set of gauge-equivalence classes  $\mathcal{A}/\mathcal{G}$ ) and a configuration  $T[\alpha]$  (with  $\alpha$  now ranging over all loops in the underlying manifold), modulo certain constraints among individual values  $T(\alpha)$ . The existence of constraints implies the loop functional  $T[\alpha]$  is not arbitrary, but has to satisfy a number of conditions which ensure it is the trace of the path-ordered product of matrices belonging to a representation of a particular gauge group [13], in our case the fundamental representation of  $SU(2)$ . These conditions are sometimes called Mandelstam constraints [14]. For the gauge group  $SU(2)$ , they are essentially summarized by the equation

$$T(\alpha)T(\beta) = T(\alpha \circ_s \beta) + T(\alpha \circ_s \beta^{-1}), \quad (1.2)$$

relating the Wilson loops of the two closed curves  $\alpha$  and  $\beta$  intersecting in a common point  $s$  to the Wilson loops of the composite curves  $\alpha \circ_s \beta$  ( $\alpha$  followed by  $\beta$ ) and  $\alpha \circ_s \beta^{-1}$  ( $\alpha$  followed by the inverse of  $\beta$ ), with  $\circ_s$  denoting as usual the composition of loops in  $s$  [3]. In other words, by working throughout with variables  $T(\alpha)$  we have got rid of problems with gauge-covariance (since all functions of the  $T(\alpha)$  will be gauge-invariant by construction), but instead have to deal with the vast overcompleteness of the set of  $T(\alpha)$ . The Wilson loops are not independent, but obey constraints non-linear in  $T$ , eq. (1.2), and similar conditions for more complicated intersecting and overlapping loop configurations that can be derived from eq. (1.2). These constraints lie at the heart of the gauge-invariant loop formulation of the  $SU(2)$ -gauge theory and the  $SU(2)_{\mathbb{C}}$ -formulation of general relativity. The  $SU(2)$ -loop variables give us information about a “reduced loop

space", obtained after factoring out by the equivalence relations given by the Mandelstam constraints (1.2). Little about the structure of this reduced space is known yet but clearly necessary for understanding the physical content of theories formulated in the loop approach.

This paper deals with the classical solution of the SU(2)-constraints (1.2). First a set of relations is derived which enable one to maximally reduce the redundancy implied by the Mandelstam constraints for configurations of  $n$  loops based at a common point. This is relevant for the continuum description of both Yang–Mills theory and gravity (for example, the search of solutions to the hamiltonian constraint in the latter [15]). Then, as an application, a new description of the reduced configuration spaces of the SU(2)-lattice gauge theory in two, three and four dimensions in terms of a set of independent Wilson loops is given. The main applications I have in mind are the three- and four-dimensional lagrangian (euclidean) theories [16] and their hamiltonian versions in  $2 + 1$  and  $3 + 1$  dimensions (with continuous time variable) [10]. It is remarkable that such a description (in terms of the Wilson loops of a subset of small lattice loops) exists at all, and that it is completely homogeneous on the hypercubic lattice (i.e. the set of independent loops is mapped onto itself under lattice translations). Note that there is also a lattice version of pure gravity in the new formulation [17], to which my results naturally apply.

The organization of the paper is as follows. Sect. 2 describes the reduction of classical loop variables  $T(\alpha\beta\gamma \cdots)$ , depending on loops  $\alpha, \beta \cdots$ , composed at a point, to functions of loop variables depending on only one or two loops. A set of independent loop variables on the  $N \times N$ -lattice is introduced in sect. 3, together with their main properties. As an illustration, the loop variables associated with the  $1 \times 1$ -lattice are given explicitly. Sect. 4 contains the generalization of this description of the reduced configuration space to the three- and four-dimensional lattice, which is relatively straightforward. The paper concludes with some comments on the relevance and possible further developments of the results presented. Appendix A contains the transformation of the  $T$ -variables to a more convenient set, called the  $L$ -variables, and the derivation of some identities appearing in the main text.

## 2. The reduction of the classical loop variables

Consider a set of  $n$  SU(2)-matrices  $U_{\alpha_i}$ ,  $i = 1, \dots, n$ , in the fundamental representation (or, more generally, of  $n$  complex unimodular  $2 \times 2$ -matrices). We will be interested in the traces of products of these matrices,  $\text{Tr}(U_{\alpha_1} U_{\alpha_2} \cdots U_{\alpha_k})$ . The present work is motivated by the new loop approach to canonical gravity and (lattice) gauge theory where a matrix  $U_\alpha$  arises as the holonomy

$$(U_\alpha)_A^B = \left( P \exp \oint_\alpha A \right)_A^B, \quad (2.1)$$

i.e. the path-ordered exponential of the integral of a connection one-form  $A$  over a closed loop  $\alpha$  lying in some three-manifold or on a hypercubic lattice, hence I will use loop language throughout.

The trace function plays a special role because of its invariance under gauge transformations  $U_\alpha \mapsto g U_\alpha g^{-1}$ ,  $g \in SU(2)$ . The quantities  $\text{Tr}(U_{\alpha_1} \cdots U_{\alpha_k})$  are not functionally independent, but obey non-linear constraints of the form

$$\text{Tr } U_\alpha \text{ Tr } U_\beta = \text{Tr } U_\alpha U_\beta + \text{Tr } U_\alpha U_\beta^{-1}, \quad (2.2)$$

as can easily be verified. These constraints are well-known special cases of identities holding for traces of complex  $n \times n$ -matrices, among physicists sometimes known as generalized Mandelstam constraints (see ref. [18] and references therein). In the loop approach they translate into algebraic constraints on the  $T$ -variables,  $T(\alpha) := \text{Tr } U_\alpha$ , leading to the relations (1.2) of the introduction.

Whenever intersecting-loop configurations become important, the resulting redundancy of  $T$ -variables has to be dealt with in some form. Up to now, the problem of imposing the set of conditions (1.2) has usually been left to the quantum theory, where quantum analogues of (1.2) have to be imposed as operator conditions on wave functions [3,8] or incorporated otherwise into the quantum representation [5,19]. In view of the central importance of the identities (1.2) one would like to have a description of the space of classical  $T$ -variables in terms of a set of independent variables, obtained by solving these constraints.

For the restricted case of loop configurations with at most two loops intersecting in one point, this was done in ref. [7], leading to a set of so-called  $L$ -variables. In the following I will extend this result to configurations with  $n$  loops intersecting in one point. In this section I will show how to reduce maximally the set of trace variables and how to express any  $\text{Tr}(U_{\alpha_1} \cdots U_{\alpha_k})$  as functions of variables  $\text{Tr}(U_{\alpha_i})$  and  $\text{Tr}(U_{\alpha_i} U_{\alpha_j})$  alone, i.e. on trace variables depending on a single matrix  $U$  or a pair of  $U$ 's.

Consider a set of  $n$  oriented loops  $\alpha_i$ ,  $i = 1, \dots, n$ , (called *basic* loops) all intersecting in a common point  $s$ , and all loops (called *composite* loops) that can be obtained by composing these loops and their inverses  $\alpha_i^{-1}$  in  $s$ . (A basic loop will also be considered as a special case of a composite loop.) The associated set of  $T$ -variables consists of all expressions  $T(\alpha_{i_1}^{\pm 1} \circ_s \alpha_{i_2}^{\pm 1} \circ_s \cdots \circ_s \alpha_{i_k}^{\pm 1}) := T(\mu)$ , with arbitrarily long but finite “products” of basic loops and their inverses as arguments. From now on the symbols  $\circ_s$  in expressions with composite loops will be omitted. The aim will be to rewrite  $T(\mu)$  in a series of steps, using algebraic identities derived from equation (1.2).

(i) Reading eq. (1.2) as

$$T(\alpha\beta^{-1}) = -T(\alpha\beta) + T(\alpha)T(\beta), \quad (2.3)$$

with  $\beta$  a positively oriented basic loop and  $\alpha$  any composite loop, we can successively eliminate all inverse basic loops and rewrite  $T(\mu)$  as a sum of products of  $T$ 's depending only on positively oriented basic loops.

(ii) Given an expression  $T(\alpha\beta\gamma \cdots)$  with  $\alpha, \beta, \gamma, \dots$  positively oriented loops, we can eliminate all multiple occurrences of loops by using

$$T(\alpha\beta\alpha\gamma) = T(\alpha\beta)T(\alpha\gamma) + T(\beta\gamma) - T(\beta)T(\gamma),$$

in particular,

$$T(\alpha^2\beta) = T(\alpha)T(\alpha\beta) - T(\beta),$$

$$T(\alpha^2) = T(\alpha)^2 - 2, \quad (2.4)$$

(where we may take  $\alpha$  to be a basic loop and  $\beta, \gamma$  composite), which can easily be derived from (1.2). This means we can rewrite  $T(\mu)$  as a sum of products of  $T$ 's, each depending on at most  $n$  distinct basic loops.

(iii) Note furthermore that the order in which the basic loops occur in such a term  $T$  is not important, since the variable  $T(\alpha\beta\gamma)$  is not independent of  $T(\beta\alpha\gamma)$  with  $\alpha$  and  $\beta$  permuted

$$\begin{aligned} T(\alpha\beta\gamma) &= -T(\beta\alpha\gamma) + T(\alpha)T(\beta\gamma) + T(\beta)T(\alpha\gamma) + T(\gamma)T(\alpha\beta) \\ &\quad - T(\alpha)T(\beta)T(\gamma). \end{aligned} \quad (2.5)$$

Hence we may bring the positively oriented basic loops appearing as arguments of any  $T$  into some given order (for example, according to their label  $i$ ).

(iv) We can re-express any  $T$  depending on four or more basic loops as a sum of products of  $T$ -variables depending on at most three basic loops, using the identity

$$\begin{aligned} T(\alpha\beta\gamma\delta) &= \frac{1}{2}(T(\alpha)T(\beta\gamma\delta) + T(\beta)T(\alpha\gamma\delta) + T(\gamma)T(\alpha\beta\delta) + T(\delta)T(\alpha\beta\gamma) \\ &\quad - T(\alpha\beta)T(\gamma)T(\delta) - T(\beta\gamma)T(\alpha)T(\delta) - T(\gamma\delta)T(\alpha)T(\beta) \\ &\quad - T(\alpha\delta)T(\beta)T(\gamma) + T(\alpha\beta)T(\gamma\delta) - T(\alpha\gamma)T(\beta\delta) \\ &\quad + T(\alpha\delta)T(\beta\gamma) + T(\alpha)T(\beta)T(\gamma)T(\delta)). \end{aligned} \quad (2.6)$$

Like all the previous identities this one holds for arbitrary loops  $\alpha, \beta, \gamma, \delta$  and in particular for basic loops. One derives equation (2.6) by adding to the expression  $T(\alpha\beta\gamma\delta) + T(\alpha^{-1}\beta\gamma\delta) = T(\alpha)T(\beta\gamma\delta)$  (obtained from (i)) an expression for the difference  $T(\alpha\beta\gamma\delta) - T(\alpha^{-1}\beta\gamma\delta)$  in terms of  $T$ 's depending on at most three loops, obtained by the combined use of (i), (iii) and the identity  $T(\alpha\delta^{-1}\gamma^{-1}\beta^{-1}) = T(\alpha^{-1}\beta\gamma\delta)$ . Note that the right-hand side of (2.6) has the correct invariance

property under cyclic permutation of the loops  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ . Hence we can express any  $T(\mu)$  as a sum of products of  $T$ 's depending on one, two or three distinct basic loops (of which there are  $n$ ,  $\frac{1}{2}n(n-1)$  and  $\frac{1}{2}n(n-1)(n-2)$  respectively).

(v) Any  $T$  depending on three basic loops is functionally dependent on  $T$ 's depending on only one or two basic loops, by virtue of the identity

$$\begin{aligned} L(\alpha\beta\gamma)^2 + L(\alpha\beta)L(\alpha\gamma)L(\beta\gamma) + \frac{1}{2}L(\alpha^2)L(\beta^2)L(\gamma^2) \\ - \frac{1}{2}L(\alpha\beta)^2L(\gamma^2) - \frac{1}{2}L(\alpha\gamma)^2L(\beta^2) - \frac{1}{2}L(\beta\gamma)^2L(\alpha^2) = 0. \end{aligned} \quad (2.7)$$

The derivation of this expression, using the SU(2)-identities, is nontrivial and can be found in appendix A. This and the following identity have been written in terms of  $L$ -variables which simplifies their form enormously. The relation between  $L$ - and  $T$ -variables is straightforward and also given in appendix A, together with the equivalents of eqs. (2.7) and (2.8) in terms of  $T$ -variables. From eq. (2.7) we conclude that the knowledge of all  $L$ 's ( $T$ 's) depending on one or two basic loops determines any  $L(\alpha\beta\gamma)$  ( $T(\alpha\beta\gamma)$ ) up to a sign.

(vi) However, the set of the  $n$  variables  $L(\alpha)$  ( $T(\alpha)$ ) and the  $\frac{1}{2}n(n-1)$  variables  $L(\alpha\beta)$  ( $T(\alpha\beta)$ ) is still redundant. There is one final set of conditions that can be obtained from eq. (1.2), namely

$$\begin{aligned} L(\alpha\beta)^2L(\gamma\delta)^2 + L(\alpha\gamma)^2L(\beta\delta)^2 + L(\alpha\delta)^2L(\beta\gamma)^2 \\ - L(\alpha\beta)^2L(\gamma^2)L(\delta^2) - L(\alpha\gamma)^2L(\beta^2)L(\delta^2) - L(\alpha\delta)^2L(\beta^2)L(\gamma^2) \\ - L(\beta\gamma)^2L(\alpha^2)L(\delta^2) - L(\beta\delta)^2L(\alpha^2)L(\gamma^2) - L(\gamma\delta)^2L(\alpha^2)L(\beta^2) \\ + 2L(\alpha^2)L(\beta\gamma)L(\beta\delta)L(\gamma\delta) + 2L(\beta^2)L(\alpha\gamma)L(\alpha\delta)L(\gamma\delta) \\ + 2L(\gamma^2)L(\alpha\beta)L(\alpha\delta)L(\beta\delta) + 2L(\delta^2)L(\alpha\beta)L(\alpha\gamma)L(\beta\gamma) \\ - 2L(\alpha\beta)L(\beta\gamma)L(\gamma\delta)L(\alpha\delta) - 2L(\alpha\gamma)L(\beta\delta)L(\alpha\delta)L(\beta\gamma) \\ - 2L(\alpha\gamma)L(\beta\delta)L(\alpha\beta)L(\gamma\delta) + L(\alpha^2)L(\beta^2)L(\gamma^2)L(\delta^2) = 0, \end{aligned} \quad (2.8)$$

whose derivation can also be found in appendix A. Because the conditions of this form are themselves not independent of each other, the elimination of this last redundancy of variables is non-trivial and will not be discussed here. The relations (2.3–2.6) are well known and have been included for the sake of convenience and completeness, whereas the relations (2.7) and (2.8) to my knowledge have not appeared in the literature.

### 3. Independent loop variables on the lattice

Our setting in this and the following section will be the  $SU(2)$ -gauge theory without fermions on a hypercubic lattice with periodic boundary conditions. We will introduce sets of independent, gauge-invariant variables  $T(\alpha)$  on the configuration spaces of these theories, given by the traces of holonomies  $U_\alpha$  (in the fundamental representation) of certain closed paths  $\alpha$  on the lattice, made up of some sequence of oriented links  $l_1 l_2 \cdots l_n$

$$T(\alpha) = \text{Tr } U_\alpha := \text{Tr}(U_{l_1} \cdots U_{l_n}) \quad l_i \in \alpha. \quad (3.1)$$

Such loop variables have been widely used in the literature since they constitute natural variables on the reduced configuration space of any pure gauge theory (i.e. after factoring out the gauge-equivalence classes). The hamiltonian loop approach advocated by Rovelli and Smolin [3,8] and independently by Gambini and Trias [5], to which our results directly apply, intends to be more radical: the theory is exclusively based on gauge-invariant trace variables and gauge-covariant quantities never appear. For example, in the quantum theory not only gauge-invariant operators like  $\overline{\text{Tr } U_p}$  (as potential term in the quantum hamiltonian;  $p$  denoting a plaquette) are employed, but also wave functions are labelled by loop configurations in a gauge-invariant manner. That it is indeed possible to formulate the  $SU(N)$  lattice gauge theory in this way was shown in ref. [6,8] and corroborated by computational results of Brüggmann for the  $SU(2)$  case on a  $4 \times 4$ -spatial lattice [9]. The results derived in this paper are also relevant to the lagrangian formulation, where an appropriate quantization however still has to be found.

The objection often raised against such a loop approach, both on the lattice and in the continuum, is the overcompleteness of the trace variables which was the subject of sect. 2 (see e.g. ref. [12]). For example, in ref. [9] a linearly independent subset of wave functions is selected by a computer algorithm from a truncated set of all wave functions in order to eliminate this redundancy, but this clearly puts a limit on computations with larger lattices and does not particularly improve our understanding of the physical contents of the lattice loop variables, or of the action of the hamiltonian. I will tackle the overcompleteness problem already at the classical level. The results presented in this and the following section should be regarded as a first step towards reconstructing classical and quantum gauge theory (and possibly gravity) in terms of gauge-invariant loop variables from scratch, without relying on a functional transform from usual formulations (as used, for example, in ref. [8]).

Mandelstam constraints exist of course for any gauge group, but have a different form for different groups. For this reason it is not straightforward to generalize the results presented here to other  $SU(N)$ -gauge groups or to perform the large- $N$  limit; already for  $SU(3)$  the analogous treatment will be much more involved, but

presumably still possible. In the following we will introduce a set of independent loop variables (3.1) in the description of hamiltonian  $SU(2)$ -lattice gauge in  $2 + 1$  dimensions, i.e. with loops lying on a spatial  $N \times N$  lattice. The generalization to the three- and four-dimensional lattice is surprisingly easy and contains nothing really new. However, since the choice of independent loop variables is quite different from the two-dimensional case, it is discussed separately in sect. 4.

Unfortunately the results of sect. 2 cannot be used immediately to construct a set of independent loop variables on the lattice. To apply the reduction procedure, one has to choose a base point  $O$  on the lattice, and all loops must go through and be composed at  $O$ . This is unphysical or at least inconvenient because it breaks the translational symmetry of the periodic lattice. One can give an algorithm for expressing any  $T(\mu)$ , with  $\mu$  a contractible loop on the lattice, in terms of variables  $T(\alpha)$  and  $T(\alpha\beta)$ , where  $\alpha$  and  $\beta$  are basic plaquette loops based and composed at  $O$  (basic plaquette loops are going round one of the  $N^2$  plaquettes with some canonically chosen “spike” [9] connecting them with the origin.) However, there is no obvious way of selecting an independent subset from this restricted set, hence I will proceed slightly differently.

In the usual Hamiltonian  $SU(2)$ -lattice gauge theory à la Kogut and Susskind [10] on a  $N \times N$  lattice with periodic boundary conditions, we know how to count the number of physical degrees of freedom. It is given by

$$\begin{aligned} & (\text{number of links} - \text{number of sites}) \times \dim(SU(2)) \\ &= (2N^2 - N^2) \times 3 = 3N^2, \end{aligned} \quad (3.2)$$

where one has used the fact that the reduced configuration space is the quotient space

$$Q_{\text{red}} = \frac{\times_l SU(2)_l}{\times_s SU(2)_s}, \quad (3.3)$$

with  $\times_l$  and  $\times_s$  denoting the product over all links and sites respectively, and that in a generic point of the configuration space  $Q = \times_l SU(2)_l$  the gauge group  $\times_s SU(2)_s$  has vanishing isotropy group, i.e. the action is essentially free.

Let us now introduce a special subset of loop variables  $T$  on the  $N \times N$  lattice. We label the  $N^2$  lattice plaquettes by two natural numbers  $i$  and  $j$ ,  $1 \leq i, j \leq n$ , according to their position on the lattice, with the identifications  $i + n \equiv i$  and  $j + n \equiv j$ . With each plaquette  $(ij)$  we associate the three trace variables

$$\begin{aligned} T(\gamma_{ij}) &:= \text{Tr } U_{\gamma_{ij}} \\ T(\gamma_{ij} \circ \gamma_{i+1,j}) &:= \text{Tr } U_{\gamma_{i,j}} U_{\gamma_{i+1,j}} \\ T(\gamma_{ij} \circ \gamma_{i,j+1}) &:= \text{Tr } U_{\gamma_{i,j}} U_{\gamma_{i,j+1}}, \end{aligned} \quad (3.4)$$



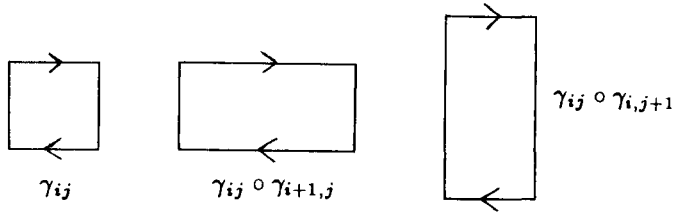


Fig. 1.

where  $\gamma_{ij}$  denotes the closed loop going round the plaquette  $(ij)$  with positive orientation (see fig. 1).

Neighbouring plaquette loops are composed in any point of the link they share in common, the contributions coming from this link cancel anyway. This gives us exactly  $3N^2$  different loop variables for the whole lattice. If we can show they are independent they must (at least locally) give a good parametrization for the reduced configuration space, since by construction they are gauge-invariant.

In order to prove that these variables are really independent, I showed that in a generic point  $p$  of the reduced configuration space  $Q_{\text{red}}$ , their associated tangent vectors span the tangent space  $T_p Q_{\text{red}}$ . Since the loop variables are constructed from local link variables  $U_l$  in an essentially local way, sufficiently separated loop variables “do not know about each other”, and (on a sufficiently big lattice) it suffices to prove the independence in a local “neighbourhood” (consisting of 17 loop variables), to be defined below. This involves the computation of the rank of a  $24 \times 17$  matrix which I found most easy to perform with the help of a computer program, for random parameter values (corresponding to random points  $p \in Q_{\text{red}}$ ).

To obtain the neighbourhood of a given link  $l$ , consider the set of links occurring in loop variables of the type (3.4) that contain also  $U_l$  (there are 8 such loop variables); then form the set of all loop variables that can be constructed from the corresponding link holonomies alone, using (3.4) (there are 17 such loop variables). We will say that the neighbourhood contains 23 links and consists of 17 loop variables (fig. 2). (Note that in the case of SU(2) the loop variable  $T(\alpha)$  does not depend on the orientation of  $\alpha$ .)

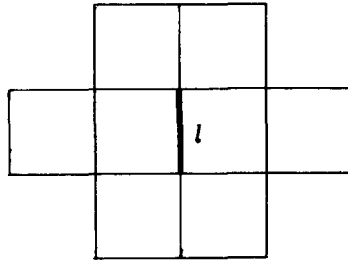


Fig. 2.

To check the independence of these 17 loops variables I used the following method.

(a) Choose a maximal gauge-fixing of links, i.e. gauge-fix a maximal tree [20] (the final result is of course independent of this gauge-fixing).

(b) Parametrize the  $SU(2)$ 's associated with the remaining eight (oriented) links as in (A.4). On each three-sphere use the coordinate patch with  $\alpha_1 \neq 0$ , parametrized by the independent variables  $\alpha_2$ ,  $\alpha_3$  and  $\alpha_4$ . (Certainly for weak fields points with  $\alpha_1 = 0$  are never reached, but also the choice  $\alpha_2 \neq 0$ , say, leads to the same result.)

(c) Express the 17 loop variables in terms of these 24 parameters and compute their gradients (17 vector fields in the 24-dimensional tangent space to the parameter space).

(d) Use a computer program to show that for generic values of the parameters the 17 vectors are linearly independent. This involves finding a submatrix of rank 17 of a  $24 \times 17$  matrix. I used a simple FORTRAN program with a MATH/LIBRARY subroutine for calculating determinants to check that one can actually find such submatrices. Once this result is established, it is easy to check that enlarging the neighbourhood by adding links (and thereby new loop variables (3.4)) cannot give rise to any new dependences among the new and old loop variables. For example, “glueing on” a plaquette at one of the four corners of fig. 2 amounts to adding one unfixed (and one gauge-fixed) link to the diagram, and at the same time increases the number of possible loop variables from 17 to 20. The three new loop variables are independent among themselves, and all three depend on the new link, whereas none of the 17 old loop variables depend on the added link. Hence all 20 loop variables form an independent set. The same is true if we add other plaquettes to the diagram. This completes the proof of local independence of these variables.

What might still happen is that new dependences arise through the periodicity of the lattice, i.e. when the neighbourhood or enlarged neighbourhood wraps around the whole lattice and some links become identified. This is to be compared with the case of the  $U(1)$ -gauge theory on the periodic  $N \times N$  lattice, where the plaquette loops  $T(\gamma_{ij})$  alone are locally independent in the sense just described. However, if one considers all  $N^2$  variables  $T(\gamma_{ij})$ , one finds one constraint between all of them, which means that only  $N^2 - 1$  are independent. This “global” constraint exists independently of the lattice size  $N$ . I could show by an explicit computation like the one described in (a)–(d) above that for the  $SU(2)$ -theory on the  $1 \times 1$ ,  $2 \times 2$  and  $3 \times 3$  lattice no such global constraints exist and the loop variables (3.4) are indeed independent. More details of the  $1 \times 1$  lattice are discussed below.

This strongly suggests that the loop variables (3.4) are a complete set of independent parameters for the reduced configuration space of the  $SU(2)$ -lattice

gauge theory, although so far I have not been able to find a proof for the absence of global constraints in the general case.

The set of (at least locally) independent lattice loop variables has a number of interesting properties. First of all they are almost local in the sense that individual loops do not extend over more than two plaquettes. Certainly one can find independent variables depending on loops of arbitrarily big size, but the aim here was to make them as local as possible. It is remarkable that all gauge-invariant information about the configuration space of the lattice theory is contained in the traces of holonomies of loops around single plaquettes and pairs of vertically or horizontally adjacent plaquettes. Of course this does not imply that trace variables not belonging to the independent set can be easily expressed in terms of them. The importance of short loops may be the reason why a cutoff in loop length as, for example, employed in [6,9] leads to reasonable results.

With SU(2) being a compact group, the reduced configuration space  $Q_{\text{red}}$  is also compact, and all  $T$ - and  $L$ -variables may assume real values between  $-2$  and  $+2$  only. Moreover, values for the independent set of loop variables (3.4) cannot be chosen freely from this range, but have to obey certain inequalities. For two horizontally adjacent plaquettes  $(ij)$  and  $(i+1, j)$ , the inequality reads

$$0 \leq \|L(\gamma_{ij}\gamma_{i+1,j})\| \leq 2\sqrt{1 - \frac{1}{4}L(\gamma_{ij})^2} \sqrt{1 - \frac{1}{4}L(\gamma_{i+1,j})^2} \quad (3.5)$$

(using the  $L$ -variables in (3.4) for convenience) and similarly for vertically adjacent pairs of lattice plaquettes. This follows immediately from the definition of these variables. Points for which equality holds, i.e.

$$\|L(\gamma_{ij}\gamma_{i+1,j})\| = 2\sqrt{1 - \frac{1}{4}L(\gamma_{ij})^2} \sqrt{1 - \frac{1}{4}L(\gamma_{i+1,j})^2}, \quad (3.6)$$

lie on the edge of the reduced configuration space. In these points the three loop variables  $L(\gamma_{ij})$ ,  $L(\gamma_{i+1,j})$  and  $L(\gamma_{ij}\gamma_{i+1,j})$  are no longer independent and neither are their associated tangent vectors in this point. This happens for configurations with  $\gamma_{ij} \times \gamma_{i+1,j} = 0$  which, according to formula (A.6), is a gauge-invariant statement. Particular cases are  $U_{\gamma_{ij}} = \mathbb{1}$ , i.e.  $L(\gamma_{ij}) = 2$ , or  $U_{\gamma_{i+1,j}} = \mathbb{1}$ . The same is true if we restrict the holonomies  $U$  to lie in the U(1)-subgroup of SU(2) defined by all matrices (A.4) with vanishing off-diagonal entries, because then  $L(\alpha\beta)$  can be computed from  $L(\alpha)$  and  $L(\beta)$ . These degeneracies of the parametrization (3.4) were of course to be expected on general grounds. The degenerate configurations provide convenient independent checks for the correctness of the determinant programs.

The use of  $L$ -variables is also particularly suited to the weak-field limit. In this limit holonomies are close to the unit matrix, which implies that  $\alpha_1$  is close to 1

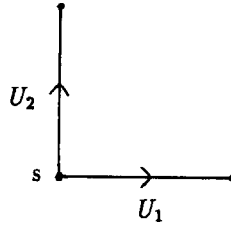


Fig. 3.

and  $\alpha_2$ ,  $\alpha_3$  and  $\alpha_4$  are of order  $\epsilon$ , with  $\epsilon$  small. Then  $L(\alpha\beta)$  is of order  $\epsilon^2$  and negligibly small compared to  $L(\alpha)$  and  $L(\beta)$ , and the theory looks “almost abelian”.

As an illustration, let us treat the  $1 \times 1$  lattice explicitly. There are only two links,  $U_1$  and  $U_2$  (fig. 3), and no gauge-fixing is possible. Under an  $SU(2)$ -gauge transformation  $g$  on the site  $s$  they transform according to  $(U_1, U_2) \rightarrow (gU_1g^{-1}, gU_2g^{-1})$ . For generic configurations the action of the gauge group  $SU(2)$  is free and hence the dimension of  $Q_{\text{red}}$  is three. The gauge-invariant loop variables (3.4) for this lattice are

$$\begin{aligned} & \text{Tr } U_2 U_1 U_2^{-1} U_1^{-1}, \\ & \text{Tr } U_2 U_1 U_1 U_2^{-1} U_1^{-1} U_1^{-1}, \\ & \text{Tr } U_2 U_2 U_1 U_2^{-1} U_2^{-1} U_1^{-1}. \end{aligned} \quad (3.7)$$

The proof of the independence of these variables can be reduced to the proof of the independence of the variables  $\text{Tr}(U_1)$ ,  $\text{Tr}(U_2)$  and  $\text{Tr}(U_1 U_2)$  (using the steps outlined in sect. 1), which is straightforward. Another thing one can show is that if  $\text{Tr}(U_1) = \text{Tr}(U_1')$ ,  $\text{Tr}(U_2) = \text{Tr}(U_2')$  and  $\text{Tr}(U_1 U_2) = \text{Tr}(U_1' U_2')$ , then one can always find a transformation  $g \in SU(2)$  such that  $U_1' = gU_1g^{-1}$  and  $U_2' = gU_2g^{-1}$ . This implies that the trace-variables indeed capture *all* the gauge-invariant information of the system. One could think of a different set of loop variables to describe the  $1 \times 1$  theory, for example,

$$\begin{aligned} & \text{Tr}(U_2 U_1 U_2^{-1} U_1^{-1}), \\ & \text{Tr}(U_2 U_2 U_1 U_2^{-1} U_2^{-1} U_1^{-1}), \\ & \text{Tr}(U_2 U_2 U_2 U_1 U_2^{-1} U_2^{-1} U_2^{-1} U_1^{-1}), \end{aligned} \quad (3.8)$$

but one immediately verifies that these are nowhere independent and hence do not provide a good parametrization for  $Q_{\text{red}}$ .

#### 4. Generalization to three and four dimensions

A key question is whether the results of sect. 3 can be generalized to higher dimensions. In the following I will show it is indeed possible to find a subset of independent loop variables on the three- and four-dimensional hypercubic lattice, and – as was the case in two dimensions – there is a surprisingly simple choice of such a subset.

We first discuss the  $N \times N \times N$  lattice and later generalize to four dimensions. The number of physical degrees of freedom for the pure gauge system is  $6N^3$ , namely 6 for each unit cube. The loops will be chosen small and thereby their corresponding variables as localized as possible. I will associate six independent loop variables with every unit cube and illustrate this choice with the help of the  $1 \times 1 \times 1$  lattice with periodic boundary conditions (fig. 4). The variable  $U_i$  denotes the  $SU(2)$  matrix associated with the oriented link in positive  $i$ -direction. The preferred set of loops consists of three plaquette-loops of link length 4 and three loops of link length 6, and their corresponding trace variables are given explicitly by

$$\begin{aligned}
 T(\alpha_1) &:= \text{Tr } U_2 U_3 U_2^{-1} U_3^{-1}, \\
 T(\alpha_2) &:= \text{Tr } U_1 U_3 U_1^{-1} U_3^{-1}, \\
 T(\alpha_3) &:= \text{Tr } U_1 U_2 U_1^{-1} U_2^{-1}, \\
 T(\alpha_1 \alpha_2) &:= \text{Tr } U_3 U_2 U_1 U_3^{-1} U_1^{-1} U_2^{-1}, \\
 T(\alpha_2 \alpha_3) &:= \text{Tr } U_1 U_3 U_2 U_1^{-1} U_2^{-1} U_3^{-1}, \\
 T(\alpha_1 \alpha_3) &:= \text{Tr } U_2 U_1 U_3 U_2^{-1} U_3^{-1} U_1^{-1},
 \end{aligned} \tag{4.1}$$

using the notation  $\alpha_i$  for the loop going around the plaquette perpendicular to the  $i$ -direction, and a suggestive notation for the longer loops extending over two plaquettes. Note that there are three different kinds of loops of link length 6 we could have used to supplement the plaquette loops: the L-shaped loops (for

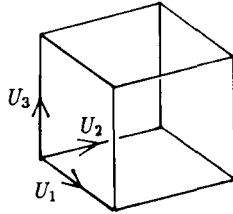


Fig. 4.

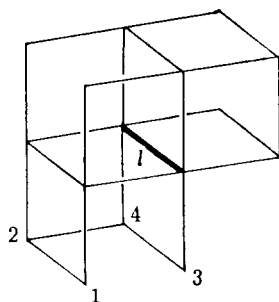


Fig. 5.

example, along the link sequence  $l_1 l_3 l_2 l_1^{-1} l_2^{-1} l_3^{-1}$  adopted in (4.1), the “unbent” loops of type  $l_1 l_3 l_3 l_1^{-1} l_3^{-1} l_3^{-1}$  and “zigzag loops” of type  $l_1 l_3 l_2 l_1^{-1} l_3^{-1} l_2^{-1}$ . Loops of the first kind were chosen because they are more local than the unbent loops and can be understood as products of just two plaquette loops, which will be important in the construction of the reduced configuration space later. Note that in the  $U(1)$  theory the number of physical degrees of freedom is 2 per unit cube and hence  $T(\alpha_1)$ ,  $T(\alpha_2)$  and  $T(\alpha_3)$  by themselves would not be independent variables. This just restates the well-known fact that in the abelian theory the Wilson plaquette sitting on the six faces of a cube are not independent [16]. We emphasize that the same statement is not true for the non-abelian theory.

The loop variables for the  $N \times N \times N$  lattice are simply given by a copy of the set (4.1) for each unit cube of the lattice (now of course without periodic boundary conditions for individual unit cubes). To show the independence of the loop variables of the loop variables (4.1), I used exactly the same method as in two dimensions, with the following changes: the neighbourhood of a link  $l$  contains now 25 links and consists of 20 loop variables (fig. 5). The maximal gauge-fixed tree consists of 15 links and hence leaves 10 links unfixed. The independence of the 20 loop variables contained in the neighbourhood could easily be established by finding submatrices of rank 20 of the relevant  $30 \times 20$  matrix. The next step consists again in showing that by “glueing on” new links to the local neighbourhood of fig. 5 (and thereby creating new loop variables of type (4.1)) no dependence between the loop variables can arise. Let us illustrate a typical step in the proof by an explicit example.

In fig. 5, glue on the link going from site 1 to site 3. This link must remain unfixed, since the gauge-fixed before adding the link was assumed to be given by a maximal tree. This enlarges the neighbourhood by two plaquette-loop variables, one of type  $T(\alpha_1)$  and one of type  $T(\alpha_3)$ . Those two variables are independent of each other, being a subset of the neighbourhood of the newly added link. On the other hand, both are independent of all other loop variables, because none of these old variables depend on the new link. Therefore the enlarged neighbourhood consists now of 22 independent loop variables. We leave it as an exercise to the

reader to prove that a further enlargement of the neighbourhood is possible by glueing on new links in steps like the one just described.

Let us now consider the generalization of the above framework to the hypercubic lattice in four dimensions. Note that in three dimensions the three independent directions could be treated on an equal footing as far as the choice of the set of independent loop variables was concerned. In four dimensions the number of physical degrees of freedom per unit cell is 9, and a similar construction is not possible. As a result, there naturally emerges a preferred direction, which we shall call the time direction. It turns out to be sufficient to supplement the three-dimensional “space-like” loop variables with three types of “time-like” loop variables to arrive at a set of independent variables in four dimensions. For the  $1 \times 1 \times 1 \times 1$  unit cell (obtained by adding links  $U_0$  in fig. 4) this amounts to adding the three loop variables

$$\begin{aligned} T(\delta_1) &:= \text{Tr } U_0 U_1 U_0^{-1} U_1^{-1}, \\ T(\delta_2) &:= \text{Tr } U_0 U_2 U_0^{-1} U_2^{-1}, \\ T(\delta_3) &:= \text{Tr } U_0 U_3 U_0^{-1} U_3^{-1}, \end{aligned} \tag{4.2}$$

to the set (4.1), with  $\delta_i$  denoting the corresponding time-like plaquette loops. Due to this anisotropic choice, the neighbourhoods of a time-like and a space-like link look very different. We will therefore consider the neighbourhood of a pair of a space-like link  $U_i$  and a time-like link  $U_0$  emerging from the same site, and repeat the procedure outlined for the three-dimensional case. This “combined” neighbourhood consists of 42 links and 27 loop variables (20 space-like of type (4.1) and 7 time-like of type (4.2)). It is straightforward to generalize the three-dimensional gauge-fixed maximal tree (25 of the 42 links can be fixed) and the proof of independence of the loop variables contained in the neighbourhood (one profitably works with the three-dimensional spatial projections of the four-dimensional link and loop configurations). Also the enlargement of the neighbourhood by glueing on links does not present any difficulties.

In order to obtain a simple description of the physical, reduced configuration space  $Q_{\text{red}}$ , it is useful to make a coordinate transformation of the double-plaquette variables, i.e. the six-link loop variables  $T(\alpha_i \alpha_j)$  to  $L(\alpha_i \alpha_j)$  according to formula (A.1). The space  $Q_{\text{red}}$  is then the cartesian product of  $3N^3$  plaquette-loop variables  $T(\alpha_i)$  and the  $3N^3$  variables  $L(\alpha_i \alpha_j)$  (on the four-dimensional lattice there are  $3N^4$   $T(\alpha_i)$ ,  $3N^4$   $L(\alpha_i \alpha_j)$  plus the  $3N^4$  time-like plaquette variables  $T(\delta_i)$ ). The variables  $L(\alpha_i \alpha_j)$  are again subject to inequalities of the form (3.5), with  $\gamma_{ij}$ ,  $\gamma_{i+1,j}$  substituted by  $\alpha_i$ ,  $\alpha_j$ . On the boundary of  $Q_{\text{red}}$  (considering only the three-dimensional theory for the moment), i.e. in points where the equality (3.6) holds for one or more pairs of plaquettes  $\alpha_i$  and  $\alpha_j$ , the parametrization becomes

singular and the dimension of  $TQ_{\text{red}}$  is less than  $6N^3$ . So far, no such singular points have been discovered in the interior of  $Q_{\text{red}}$ . Note that with our particular choice of loop variables a given spatial plaquette  $\alpha$  occurs twice in some  $L$ -variable argument.

This completes the description of the independent loop variables in three and four dimensions. Again, in principle dependences between the loop variables could arise when one starts identifying some of the link and loop variables while enlarging a neighbourhood, because of the periodicity of the boundary conditions. The absence of such “global constraints” has been verified by explicit computation for the  $1^3$  and  $1^4$  lattices. This can be taken as an indication of their absence in general, although – like in the two-dimensional case – we do not as yet have an explicit proof of this fact.

## 5. Conclusions

For pure  $SU(2)$ -lattice gauge theory in two, three and four dimensions, there is complete description of the reduced configuration space, given in terms of  $3N^2$ ,  $6N^3$  and  $9N^4$  independent loop variables respectively (provided one can rule out global constraints for lattice sizes bigger than  $N = 3$  in two and bigger than  $N = 1$  in three and four dimensions). This is remarkable, since from the structure of the quotient (3.3) it is not clear that such a description should exist at all. What may be surprising is the simplicity of our solutions for the independent loop variables. They are localized around individual unit cells (two unit cells in the two-dimensional case) and describe the lattice in a very homogeneous way. This is in contrast with the usual, maximally gauge-fixed description in terms of the gauge-covariant link variables  $U_l$ . A direct comparison between this and our gauge-invariant approach is not straightforward, because the gauge-fixing in the former is never complete and the form of a maximal tree is not invariant under lattice translations, unlike the set of independent loop variables, which is symmetric under lattice translations, i.e. does not distinguish any particular link on the lattice. From the point of view of the intrinsically gauge-invariant formulation the gauge-covariant description looks somewhat indirect and awkward. Finding independent lattice variables is a non-trivial result, since it amounts to solving a highly coupled set of non-linear equations (1.2).

There are in principle two ways of quantizing the  $SU(2)$ -lattice gauge theory, given the independent loop variables. In a hamiltonian approach, one has to supplement the configuration variables by appropriate momentum variables. This could be done along the lines proposed, for example, in ref. [8], where one uses gauge-invariant generalizations  $T^n(\alpha)$ ,  $n \geq 0$ , of the variables  $T(\alpha)$ , with  $n$  conjugate momenta  $E$  inserted into the traces of the holonomies. The central problem is then to rewrite the classical Poisson algebra formed by the  $T^n(\alpha)$  (which suffer



from the same overcompleteness as the  $T(\alpha) \equiv T^0(\alpha)$  in terms of an independent subset of the  $T^n(\alpha)$ . In the lagrangian formulation, the next step consists in finding the correct form of the usual invariant group measure  $\int[dU]$  [20] in terms of the independent  $L$ -variables, which is necessary for the computation of expectation values of observables. Work on this problem is in progress.

How the results of this paper can be exploited in the continuum formulation of Yang–Mills theory and gravity remains to be seen. Unlike on the lattice, in the continuum we do not have the concept of a smallest loop (of finite size). An interesting question is how the reduction of classical loop variables can be implemented on the representation space of the quantum theory (usually given in terms of a set of loop functionals). This is also relevant to the search of solutions to the hamiltonian constraint in quantum gravity [3,15,19]. For instance, the left-hand side of eq. (A.6), viewed as a loop wave functional [15], is annihilated by the hamiltonian constraint operator, and there are presumably more such solutions which assume a simple form in terms of the  $L$ -variables.

Many thanks to P. Lauwers for discussions about lattice gauge theory.

## Appendix A

In this appendix we introduce a different, but equivalent set of loop variables and derive the identities (2.7) and (2.8). In terms of these so-called  $L$ -variables the identities assume a much simpler form. The  $L$ -variables are totally “antisymmetric” with respect to basic loops and were already introduced in ref. [7] in a more restricted context. These particular combinations of  $T$ -variables appear in a natural way on the right-hand sides of Poisson-bracket relations among variables  $T^n$ . For positively oriented basic loops  $\alpha, \beta, \gamma, \dots$ , define

$$\begin{aligned}
 L(\alpha) &:= T(\alpha), \\
 L(\alpha\beta) &:= \frac{1}{2}(T(\alpha\beta) - T(\alpha\beta^{-1})) = T(\alpha\beta) - \frac{1}{2}T(\alpha)T(\beta), \\
 L(\alpha\beta\gamma) &:= \frac{1}{8}(T(\alpha\beta\gamma) - T(\alpha\beta^{-1}\gamma) - T(\alpha\beta\gamma^{-1}) + T(\alpha\beta^{-1}\gamma^{-1}) \\
 &\quad - T(\alpha\gamma\beta) + T(\alpha\gamma^{-1}\beta) + T(\alpha\gamma\beta^{-1}) - T(\alpha\gamma^{-1}\beta^{-1})) \\
 &= T(\alpha\beta\gamma) - \frac{1}{2}T(\alpha)T(\beta\gamma) - \frac{1}{2}T(\beta)T(\alpha\gamma) \\
 &\quad - \frac{1}{2}T(\gamma)T(\alpha\beta) + \frac{1}{2}T(\alpha)T(\beta)T(\gamma). \tag{A.1}
 \end{aligned}$$

We can invert these equations and obtain expressions for  $T(\alpha)$ ,  $T(\alpha\beta)$  and  $T(\alpha\beta\gamma)$  in terms of  $L$ -variables. From this definition follow the properties

$$\begin{aligned} L(\alpha\beta) &= -L(\alpha^{-1}\beta) = -L(\alpha\beta^{-1}) = L(\alpha^{-1}\beta^{-1}) = L(\beta\alpha), \\ L(\alpha\beta\gamma) &= -L(\alpha^{-1}\beta\gamma) = -L(\alpha\beta^{-1}\gamma) = -L(\alpha\beta\gamma^{-1}), \\ L(\alpha\beta\gamma) &= L(\beta\gamma\alpha) = L(\gamma\alpha\beta) = -L(\alpha\gamma\beta), \end{aligned} \quad (\text{A.2})$$

which in turn imply

$$\begin{aligned} L(\alpha\alpha) &= T(\alpha\alpha) - \frac{1}{2}T(\alpha)^2 = \frac{1}{2}L(\alpha)^2 - 2, \\ L(\alpha\alpha\beta) &= 0. \end{aligned} \quad (\text{A.3})$$

The  $L$ -variables have a straightforward geometrical meaning: recall the standard parametrization for SU(2)-matrices in the fundamental representation

$$U_\alpha = \begin{pmatrix} \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \end{pmatrix}, \quad (\text{A.4})$$

$\alpha_i$  real,  $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 1$ , which realizes the imbedding of SU(2) as three-sphere into  $\mathbb{R}^4$ . If  $\alpha$ ,  $\beta$  and  $\gamma$  are basic loops, the associated loop variables are

$$\begin{aligned} L(\alpha) &= 2\alpha_1, \\ L(\alpha\beta) &= -2(\alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4) = -2\boldsymbol{\alpha}_\perp \cdot \boldsymbol{\beta}_\perp, \\ L(\alpha\beta\gamma) &= -2\boldsymbol{\gamma}_\perp \cdot (\boldsymbol{\alpha}_\perp \times \boldsymbol{\beta}_\perp), \end{aligned} \quad (\text{A.5})$$

where we have introduced a convenient vectorial notation. We represent an element  $U_\alpha$  of SU(2) as a unit vector  $\boldsymbol{\alpha}$  in  $\mathbb{R}^4$  and denote the components in the 2-, 3- and 4-direction by the three-vector  $\boldsymbol{\alpha}_\perp$ .

The variable  $L(\alpha)$  just gives us twice the 1-component of the vector  $\boldsymbol{\alpha}$ . If we consider  $L$ -variables of single basic loops, the SU(2)'s associated with different loops are unrelated. However, in  $L(\alpha\beta)$  the components of both vectors get mixed, and  $L(\alpha\beta)$  is the scalar product of the two vectors  $\boldsymbol{\alpha}_\perp$  and  $\boldsymbol{\beta}_\perp$ . Hence we can understand the variables  $L(\alpha\beta)$  as specifying the “relative orientation” of the two SU(2)'s associated with the two basic loops  $\alpha$  and  $\beta$ . The vectorial notation exhibits the fact that all constraints on  $L$ - or  $T$ -variables are equivalent to certain vector product identities of vectors in  $\mathbb{R}^3$ . Another useful relation in this context is

$$L(\alpha\beta)^2 - L(\alpha^2)L(\beta^2) = -4(\boldsymbol{\alpha}_\perp \times \boldsymbol{\beta}_\perp)^2. \quad (\text{A.6})$$

The new variables will be used in the following to derive the identities (2.7) and (2.8). Note that one can obtain non-trivial identities from  $T$ -variables depending on five (or more) basic loops with the help of the 4-loop identity (2.6). For example, take a variable  $T(\alpha\beta\gamma\delta\epsilon)$  depending on five basic loops. Apply eq. (2.6) to  $T(\alpha\beta\mu\epsilon)$ , with  $\mu = \gamma \circ \delta$  treated as one loop, then substitute  $\gamma \circ \delta$  back into the resulting expression and eliminate all terms  $T$  depending on four basic loops by repeated use of eq. (2.6). Do the same for the variable  $T(\alpha\beta\gamma\lambda)$ , where  $\lambda = \delta \circ \epsilon$ . From  $T(\alpha\beta\mu\epsilon) - T(\alpha\beta\gamma\lambda) = 0$  we then obtain an identity for five basic loops in terms of  $T$ 's depending on at most three basic loops. In terms of  $L$ -variables it has the form

$$L(\alpha\beta\gamma)L(\delta\epsilon) + L(\alpha\gamma\epsilon)L(\beta\delta) - L(\beta\gamma\epsilon)L(\alpha\delta) - L(\alpha\beta\epsilon)L(\gamma\delta) = 0, \quad (\text{A.7})$$

and similar expressions one obtains by cyclic permutation of  $\alpha, \beta, \gamma, \delta$  and  $\epsilon$ . From eq. (A.7) follows a four-loop identity by setting  $\delta = \epsilon$ . In order to derive non-trivial three-loop identities, one has to start from an expression  $T(\alpha\beta\gamma\delta\epsilon\mu)$  depending on six basic loops. If we apply to the identity

$$T((\alpha\beta\gamma)\delta\epsilon\mu) - T(\alpha(\beta\gamma\delta)\epsilon\mu) = 0, \quad (\text{A.8})$$

the four-loop identity (2.6) in the manner described above, we obtain an expression containing two  $T$ 's depending on five basic loops, namely,  $T(\alpha(\beta\gamma)\epsilon\mu)$  and  $T((\beta\gamma)\delta\epsilon\mu)$ . These we reduce via eq. (2.6) as indicated by the parentheses. Thus we can rewrite eq. (A.8) in terms of  $T$ 's with at most four basic loops. This expression can then unambiguously be reduced using equation (2.6). After *a lot* of algebra one obtains the identity

$$\begin{aligned} & L(\delta\epsilon\mu)(L(\beta)L(\alpha\gamma) + L(\gamma)L(\alpha\beta)) - L(\alpha\epsilon\mu)(L(\beta)L(\gamma\delta) + L(\gamma)L(\beta\delta)) \\ & + 2L(\alpha\beta\gamma)L(\delta\epsilon\mu) - 2L(\alpha\epsilon\mu)L(\beta\gamma\delta) + L(\alpha\beta\mu)L(\delta\epsilon)L(\gamma) \\ & - L(\alpha\beta\epsilon)L(\delta\mu)L(\gamma) + L(\alpha\gamma\mu)L(\delta\epsilon)L(\beta) - L(\alpha\gamma\epsilon)L(\delta\mu)L(\beta) \\ & + L(\beta\delta\mu)L(\alpha\epsilon)L(\gamma) - L(\beta\delta\epsilon)L(\alpha\mu)L(\gamma) + L(\gamma\delta\mu)L(\alpha\epsilon)L(\beta) \\ & - L(\gamma\delta\epsilon)L(\alpha\mu)L(\beta) + L(\alpha\beta)L(\gamma\mu)L(\delta\epsilon) - L(\alpha\beta)L(\gamma\epsilon)L(\delta\mu) \\ & + L(\alpha\gamma)L(\beta\epsilon)L(\delta\mu) - L(\alpha\gamma)L(\beta\mu)L(\delta\epsilon) + L(\alpha\epsilon)L(\beta\mu)L(\gamma\delta) \\ & - L(\alpha\epsilon)L(\beta\delta)L(\gamma\mu) + L(\alpha\mu)L(\beta\delta)L(\gamma\epsilon) - L(\alpha\mu)L(\beta\epsilon)L(\gamma\delta) = 0. \end{aligned} \quad (\text{A.9})$$

Setting now  $\mu = \alpha, \delta = \beta$  and  $\epsilon = \gamma$ , we arrive at the three-loop identity (2.7). Now substitute in expression (A.9)  $(\alpha\beta\gamma\delta\epsilon\mu)$  by  $(\alpha\beta\gamma\beta\gamma\delta)$ , and add to the result again

expression (A.9), this time with the substitution  $(\alpha\beta\gamma\delta\epsilon\mu) \rightarrow (\alpha\gamma\beta\beta\gamma\delta)$ . This leads to the four-loop identity

$$\begin{aligned} L(\alpha\beta\gamma)L(\beta\gamma\delta) = & \frac{1}{2} \left( -L(\alpha\beta)L(\gamma\delta)L(\beta\gamma) - L(\alpha\gamma)L(\beta\delta)L(\beta\gamma) \right. \\ & + L(\alpha\delta)L(\beta\gamma)^2 + L(\alpha\beta)L(\gamma^2)L(\beta\delta) \\ & \left. + L(\alpha\gamma)L(\gamma\delta)L(\beta^2) - L(\alpha\delta)L(\beta^2)L(\gamma^2) \right). \quad (\text{A.10}) \end{aligned}$$

Combining formulas (2.7) and (A.10) leads to eq. (2.8). For the sake of completeness and to illustrate the usefulness of the  $L$ -variables, we finally give the expression for eqs. (2.7) and (2.8) in terms of  $T$ -variables. They are

$$\begin{aligned} & \left( T(\alpha\beta\gamma) - \frac{1}{2}T(\alpha)T(\beta\gamma) - \frac{1}{2}T(\beta)T(\alpha\gamma) - \frac{1}{2}T(\gamma)T(\alpha\beta) + \frac{1}{2}T(\alpha)T(\beta)T(\gamma) \right)^2 \\ & + \left( T(\alpha\beta) - \frac{1}{2}T(\alpha)T(\beta) \right) \left( T(\alpha\gamma) - \frac{1}{2}T(\alpha)T(\gamma) \right) \left( T(\beta\gamma) - \frac{1}{2}T(\beta)T(\gamma) \right) \\ & - \frac{1}{2} \left( T(\alpha\beta) - \frac{1}{2}T(\alpha)T(\beta) \right)^2 \left( \frac{1}{2}T(\gamma)^2 - 2 \right) \\ & - \frac{1}{2} \left( T(\alpha\gamma) - \frac{1}{2}T(\alpha)T(\gamma) \right)^2 \left( \frac{1}{2}T(\beta)^2 - 2 \right) \\ & - \frac{1}{2} \left( T(\beta\gamma) - \frac{1}{2}T(\beta)T(\gamma) \right)^2 \left( \frac{1}{2}T(\alpha)^2 - 2 \right) \\ & + \frac{1}{2} \left( \frac{1}{2}T(\alpha)^2 - 2 \right) \left( \frac{1}{2}T(\beta)^2 - 2 \right) \left( \frac{1}{2}T(\gamma)^2 - 2 \right) = 0. \quad (\text{A.11}) \end{aligned}$$

and

$$\begin{aligned} & T(\alpha\beta)^2 T(\gamma\delta)^2 + T(\alpha\gamma)^2 T(\beta\delta)^2 + T(\alpha\delta)^2 T(\beta\gamma)^2 \\ & + T(\alpha\beta)^2 (T(\gamma)^2 + T(\delta)^2 - T(\gamma)T(\delta)T(\gamma\delta) - 4) \\ & + T(\alpha\gamma)^2 (T(\beta)^2 + T(\delta)^2 - T(\beta)T(\delta)T(\beta\delta) - 4) \\ & + T(\alpha\delta)^2 (T(\beta)^2 + T(\gamma)^2 - T(\beta)T(\gamma)T(\beta\gamma) - 4) \\ & + T(\beta\gamma)^2 (T(\alpha)^2 + T(\delta)^2 - T(\alpha)T(\delta)T(\alpha\delta) - 4) \\ & + T(\beta\delta)^2 (T(\alpha)^2 + T(\gamma)^2 - T(\alpha)T(\gamma)T(\alpha\gamma) - 4) \\ & + T(\gamma\delta)^2 (T(\alpha)^2 + T(\beta)^2 - T(\alpha)T(\beta)T(\alpha\beta) - 4) \\ & + (T(\alpha)^2 - 4)T(\beta\gamma)T(\beta\delta)T(\gamma\delta) + (T(\beta)^2 - 4)T(\alpha\gamma)T(\alpha\delta)T(\gamma\delta) \end{aligned}$$

$$\begin{aligned}
& + (T(\gamma)^2 - 4)T(\alpha\beta)T(\alpha\delta)T(\beta\delta) + (T(\delta)^2 - 4)T(\alpha\beta)T(\alpha\gamma)T(\beta\gamma) \\
& + (T(\alpha\beta)T(\gamma\delta) + T(\alpha\gamma)T(\beta\delta))(T(\alpha)T(\delta)T(\beta\gamma) + T(\beta)T(\gamma)T(\alpha\delta)) \\
& + (T(\alpha\beta)T(\gamma\delta) + T(\alpha\delta)T(\beta\gamma))(T(\alpha)T(\gamma)T(\beta\delta) + T(\beta)T(\delta)T(\alpha\gamma)) \\
& + (T(\alpha\delta)T(\beta\gamma) + T(\alpha\gamma)T(\beta\delta))(T(\alpha)T(\beta)T(\gamma\delta) + T(\gamma)T(\delta)T(\alpha\beta)) \\
& - 2T(\alpha\beta)T(\beta\gamma)T(\gamma\delta)T(\alpha\delta) - 2T(\alpha\gamma)T(\beta\delta)T(\alpha\delta)T(\beta\gamma) \\
& - 2T(\alpha\gamma)T(\beta\delta)T(\alpha\beta)T(\gamma\delta) \\
& + (2 - T(\alpha)^2)(T(\beta)T(\gamma)T(\beta\delta)T(\gamma\delta) + T(\beta)T(\delta)T(\beta\gamma)T(\gamma\delta) \\
& + T(\gamma)T(\delta)T(\beta\gamma)T(\beta\delta)) \\
& + (2 - T(\beta)^2)(T(\alpha)T(\gamma)T(\alpha\delta)T(\gamma\delta) + T(\alpha)T(\delta)T(\alpha\gamma)T(\gamma\delta) \\
& + T(\gamma)T(\delta)T(\alpha\gamma)T(\alpha\delta)) \\
& + (2 - T(\gamma)^2)(T(\alpha)T(\beta)T(\alpha\delta)T(\beta\delta) + T(\alpha)T(\delta)T(\alpha\beta)T(\beta\delta) \\
& + T(\beta)T(\delta)T(\alpha\beta)T(\alpha\delta)) \\
& + (2 - T(\delta)^2)(T(\alpha)T(\beta)T(\alpha\gamma)T(\beta\gamma) + T(\alpha)T(\gamma)T(\alpha\beta)T(\beta\gamma) \\
& + T(\beta)T(\gamma)T(\alpha\beta)T(\alpha\gamma)) \\
& + T(\alpha)T(\beta)T(\alpha\beta)(T(\gamma)^2T(\delta)^2 - 2T(\gamma)^2 - 2T(\delta)^2 + 4) \\
& + T(\alpha)T(\gamma)T(\alpha\gamma)(T(\beta)^2T(\delta)^2 - 2T(\beta)^2 - 2T(\delta)^2 + 4) \\
& + T(\alpha)T(\delta)T(\alpha\delta)(T(\beta)^2T(\gamma)^2 - 2T(\beta)^2 - 2T(\gamma)^2 + 4) \\
& + T(\beta)T(\gamma)T(\beta\gamma)(T(\alpha)^2T(\delta)^2 - 2T(\alpha)^2 - 2T(\delta)^2 + 4) \\
& + T(\beta)T(\delta)T(\beta\delta)(T(\alpha)^2T(\gamma)^2 - 2T(\alpha)^2 - 2T(\gamma)^2 + 4) \\
& + T(\gamma)T(\delta)T(\gamma\delta)(T(\alpha)^2T(\beta)^2 - 2T(\alpha)^2 - 2T(\beta)^2 + 4) \\
& + T(\alpha)^2T(\beta)^2T(\gamma)^2 + T(\alpha)^2T(\beta)^2T(\delta)^2 \\
& + T(\alpha)^2T(\gamma)^2T(\delta)^2 + T(\beta)^2T(\gamma)^2T(\delta)^2 \\
& - T(\alpha)^2T(\beta)^2T(\gamma)^2T(\delta)^2 - 4(T(\alpha)^2 + T(\beta)^2 + T(\gamma)^2 + T(\delta)^2) + 16 = 0.
\end{aligned}$$

(A.12)

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